Time Division Multiplex Systems

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Introduction

THE idea of transmitting and receiving independent signals over a common line by means of synchronized switches at the terminals is quite old and has been used in multiplex telegraphy for many years. In general if N signal channels are to be provided over one line, the switching cycle includes N equal time intervals, one of which is allotted to each channel. Each channel is connected to the line throughout a part of its particular time interval and is disconnected throughout the remainder of the cycle. Absence of interference between the channels depends upon the fact that the channels are connected to the line throughout mutually exclusive time intervals. It is thus possible to avoid the use of channel band filters such as are necessary in carrier systems employing frequency as the basis of separation.

Application of time division multiplex methods to telephone channels has been proposed from time to time and some experiments have been made.[1,2,3,4,5,6] It is fairly evident that the concept of simple on-and-off switching giving alternately transmission and complete suppression for the signal from a particular channel on the line is inadequate for speech waves in actual telephone circuits. Imperfections in the transmission properties of the line tend to distort the wave form of the successive signal components and prolong the contribution of one signal into the time allotted for a different channel. It is the object of this paper to present a general quantitative discussion of the factors which enter into the transmission of any type of signal by a system of this kind. It has been found possible to arrive at definite criteria for such matters as the required switching frequency, the conditions to be imposed on contact time for good crosstalk suppression with economy of frequency band, and the transmission requirements which must be met by the intervening circuit to hold the interference between channels to tolerable values. The analysis leads directly to a physical viewpoint of the whole process which, to those familiar with the carrier and

[1] Patten and Minor, *U. S. Patent* 745,734, 1903.
[2] *Electrical World*, Dec. 5, 1903.
[3] Goldschmidt, *U. S. Patent* 1,087,113, Feb. 17, 1914.
[4] Poirson, *Soc. Fr. El.*, Apr. 1920.
[5] Marro, *L'Onde Electrique*, Oct. 1938.
[6] M. Cornilleau, *Revue de Telephones, Telegraphes et T. S. F.*, 13 (1935), pp. 625–643.

sideband philosophy of signal transmission, illuminates the manner in which departures from ideal amplitude and phase characteristics cause crosstalk between the several message channels. It further leads directly to other physical methods for producing and detecting a transmitted signal identical with the essential components derived in time division or switching processes.

A first step in the theoretical solution of the problem was taken by Dr. J. R. Carson, who, in an unpublished memorandum of May 25, 1920, derived quantitative relations between band width and interchannel interference in time division multiplex transmission. Applying Fourier series analysis to on-and-off switching, he showed that if the transmission medium had constant attenuation and linear phase shift for all frequencies below cutoff and no transmission of frequencies above the cutoff, the band width required for satisfactory multichannel telephony would be much wider than needed in conventional carrier methods. A further step was taken by Dr. H. Nyquist, who, in unpublished memoranda of August 24, 1936 and November 12, 1936,[7] showed that the width of band necessary may be reduced by providing a specially devised type of non-uniform transmission characteristic. In the following discussion, we shall see that a similar result can be obtained by control of the switching, and specific switching processes will be described which allow a flat transmission band of minimum width to be used.

In order to arrive at requirements which must be imposed on the various components of the system, we shall first give a theory of time division multiplex transmission in which both the switching processes and the transmission characteristic are completely general. Specific forms which give crosstalk suppression will then be discussed and effects of small departures estimated.

General Theory

We shall assume an N-channel system with a sinusoidal signal impressed on the j^{th} channel. An illustrative arrangement is shown in Fig. 1. Since the system is linear, we may represent currents and voltages by complex quantities with the understanding that the actual currents and voltages are the real components of the expressions used. Accordingly, let the signal voltage impressed on the j^{th} channel be

$$E_j(t) = E_j e^{i\omega_j t} \tag{1}$$

[7] Basic concepts used in Nyquist's analysis were included in his paper, "Certain Topics in Telegraph Transmission Theory," *A. I. E. E., Trans.*, April, 1928, pp. 617–644. Mention is also there made of the equivalence of signal shaping and equalizing in effect on reception of telegraph signals.

and let the switching between the j^{th} channel and the line at the sending end be represented by:

$$I_{sj}(t) = F_j(t)E_j(t), \tag{2}$$

where $I_{sj}(t)$ is the current flowing into the line from the j^{th} channel. The function $F_j(t)$ has the dimensions of an admittance and, in the arrangement shown in Fig. 1, is periodic in time with fundamental frequency $q = 2\pi/T$ radians per second, where T is the time occupied by one cycle of the switching operation. In the interests of economy of analysis, it is preferable for our purposes to assume for $F_j(t)$ a somewhat more general function of time

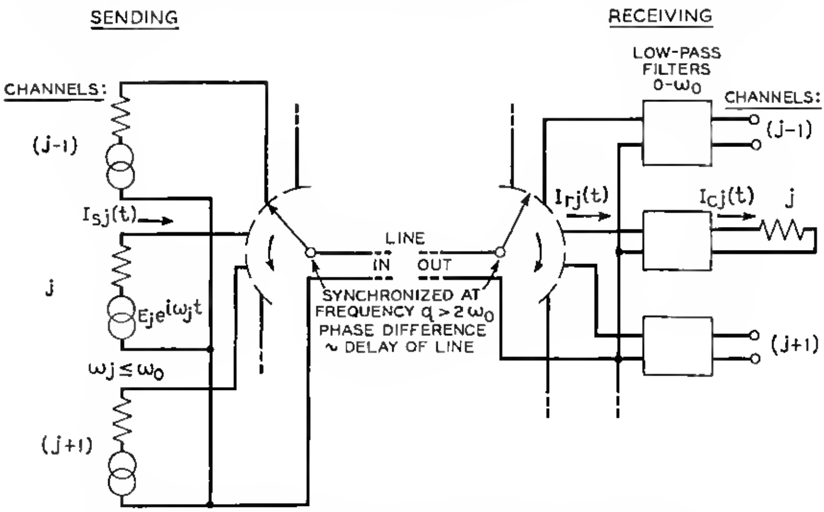

Fig. 1—Elementary arrangement for time division multiplex system

than is directly obtainable with the elementary arrangement of Fig. 1. We shall let

$$F_j(t) = \sum_{m=0}^{\infty} A_{mj} \cos\,[(\nu + mq)t - \theta_{mj}]. \tag{3}$$

To make the results applicable to Fig. 1, we merely let $\nu = 0$; then by the usual Fourier series analysis,

$$\left.\begin{aligned}
&A_{0j} = a_0/2, \qquad \left.\begin{aligned} A_{mj}^2 &= a_{mj}^2 + b_{mj}^2 \\ \tan\theta_{mj} &= b_{mj}/a_{mj} \end{aligned}\right\} m > 0 \\
&\theta_{0j} = 0, \\
&a_{mj} = \frac{2}{T}\int_{t_1}^{T+t_1} F_j(t)\cos mqt\,dt \\
&b_{mj} = \frac{2}{T}\int_{t_1}^{T+t_1} F_j(t)\sin mqt\,dt,\ t_1 \text{ arbitrary}
\end{aligned}\right\} \tag{4}$$

The wave (3) consists of the output of the circuit of Fig. 1 with all frequencies shifted by a constant amount ν radians per second; various means of accomplishing this result in the switching process will be discussed later. It is sufficient to point out here that such a shift in frequency is often desirable for optimum utilization of the transmission medium. Combining (2) and (3), we then have:

$$I_{sj}(t) = \frac{E_j}{2} \sum_{m=0}^{\infty} A_{mj} [e^{i(\nu+mq+\omega_j)t - i\theta_{mj}} + e^{-i(\nu+mq-\omega_j)t+i\theta_{mj}}] \tag{5}$$

It is clear from (5) that the result of the switching process is the production of upper and lower side frequencies from the signal on each harmonic of the switching frequency. It is also evident that if more than one signal component is superimposed, the resulting side frequencies constitute sidebands of the same nature as used in amplitude modulation systems. A significant difference between time division and amplitude modulation appears in that in the latter only one sideband or at most one pair of sidebands is transmitted, while the essential character of time division depends on the transmission of a plurality of sidebands. Thus if one pair of sidebands were selected from the output (5) by filtering, the time division process would merely be a particular way of generating the sidebands required in an amplitude modulation system.

The next step in a time division system is the transmission of the wave (5) over a line. The properties of the line in general may be specified by a complex transfer impedance, which we may express here by the ratio of open-circuit output voltage to input current:

$$E_r/I_s = Z(i\omega) \tag{6}$$

The result of applying the wave (5) to the line is then the open-circuit voltage:

$$\begin{aligned} E_{rj}(t) = \frac{E_j}{2} \sum_{m=0}^{\infty} A_{mj} Z[i(\nu + mq + \omega_j)] e^{i(\nu+mq+\omega_j)t - i\theta_{mj}} \\ + \frac{E_j}{2} \sum_{m=0}^{\infty} A_{mj} Z^*[i(\nu + mq - \omega_j)] e^{i(\nu+mq-\omega_j)t+i\theta_{mj}} \end{aligned} \tag{7}$$

In the above we have adopted the notation $Z^*(i\omega)$ to represent the conjugate of $Z(i\omega)$ and have made use of the fact that the response of a network to the applied wave $e^{-i\omega t}$ is the conjugate of the response to $e^{i\omega t}$.

At the receiving end another switching process takes place synchronously with that at the transmitting end. We shall assume that the switching process between the k^{th} channel and the line is represented by the relation

$$I_{rk}(t) = G_k(t) E_{rj}(t), \tag{8}$$

where $I_{rk}(t)$ is the current received in the k^{th} channel and $G_k(t)$ is a periodic function of time with fundamental frequency q. It is understood that j and k may be any two of the N channels. We shall express $G_k(t)$ in a manner analogous to the corresponding switching function at the transmitter, i.e.,

$$G_k(t) = \sum_{n=0}^{\infty} B_{nk} \cos\left[(\nu + nq)t - \Phi_{nk}\right] \tag{9}$$

Combining (7), (8), and (9), we find

$$I_{rk}(t) = \frac{E_j}{4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mj} B_{nk} Z[i(\nu + mq + \omega_j)]$$

$$\left(e^{i[2\nu+(m+n)q+\omega_j]t - i(\theta_{mj}+\Phi_{nk})} + e^{i[(m-n)q+\omega_j]t - i(\theta_{mj}-\Phi_{nk})}\right)$$

$$+ \frac{E_j}{4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mj} B_{nk} Z^*[i(\nu + mq - \omega_j)]$$

$$\left(e^{-i[2\nu+(m+n)q-\omega_j]t + i(\theta_{mj}+\Phi_{nk})} + e^{i[(n-m)q+\omega_j]t + i(\theta_{mj}-\Phi_{nk})}\right) \tag{10}$$

The received wave thus consists of a doubly infinite set of side frequencies involving harmonics of q. It is, however, possible to set up conditions under which the original signal may be selected and the frequencies involving the switching rate may be suppressed by filtering. If $\nu = 0$, such separation is possible provided

$$\omega_j < q/2\,, \tag{11}$$

for it then follows that a low-pass filter with cutoff frequency at $q/2$ will not pass any of the components with frequencies dependent on q. The condition (11) follows from the fact that the lowest frequency of (10) dependent on q is $q - \omega_j$, and hence we must make $q - \omega_j > \omega_j$ in order to separate ω_j from $q - \omega_j$. In other words the sidebands on adjacent harmonics must not overlap. If $\nu > 0$, the condition (11) also suffices as far as suppression of terms dependent on q are concerned, but an additional condition is required to suppress frequencies dependent on ν in the special case in which $\nu < q/2$, i.e., the case of ν less than the maximum allowable value of ω_j. For in the latter case the frequency $2\nu + (m + n)q - \omega_j$ is less than $q - \omega_j$ in the special case of $m = n = 0$. The additional condition needed is evidently either $A_{0j} = 0$ or $B_{0k} = 0$. If $\nu = 0$ or if $\nu > q/2$, this condition is unnecessary.

Assuming then that (11) is fulfilled, and that a low-pass filter with cutoff at $q/2$ is inserted in the output of each channel, we calculate for the typical channel output current:

$$I_{ek}(t) = Y_{jk} E_j e^{i\omega_j t}, \tag{12}$$

where the value of Y_{jk} is as follows:

Case 1, $\nu = 0$

$$Y_{jk} = A_{0j}B_{0k}Z(i\omega_j) + \frac{1}{4}\sum_{m=1}^{\infty} A_{mj}B_{mk}Z[i(mq + \omega_j)]e^{-i(\theta_{mj}-\Phi_{nk})} + \frac{1}{4}\sum_{m=1}^{\infty} A_{mj}B_{mk}Z^*[i(mq - \omega_j)]e^{i(\theta_{mj}-\Phi_{nk})} \quad (13)$$

Case 2, $\nu > 0$; A_{0j} or $B_{0k} = 0$ if $\nu < q/2$

$$Y_{jk} = \frac{1}{4}\sum_{m=0}^{\infty} A_{mj}B_{mk}Z[i(\nu + mq + \omega_j)]e^{-i(\theta_{mj}-\Phi_{mk})} + \frac{1}{4}\sum_{m=0}^{\infty} A_{mj}B_{mk}Z^*[i(\nu + mq - \omega_j)]e^{i(\theta_{mj}-\Phi_{mk})} \quad (14)$$

The combination of an N-channel time division multiplex system with low-pass filters in the receiving branches is thus found to be equivalent to a linear network having N pairs of input and output terminals with the transfer admittance from the j^{th} pair of input terminals to the k^{th} pair of output terminals given by Y_{jk} in (13) or (14). The transfer admittance is calculated by summing the contributions of upper and lower sidebands on harmonics of the switching frequency and is affected directly by the transmitting properties of the medium at the side band frequencies. The result we have obtained is of sufficient generality to include all cases we shall treat in this paper. We shall now proceed to specific examples.

On-and-Off Switching with Commutator

When an ideal commutator is used as a switching means, the switching functions for the N channels are identical except for a time displacement which is the same between all pairs of consecutive channels. This condition is expressed by:

$$F_j(t) = F_1[t - (j - 1)T/N] \quad (15)$$

Thus $F_1(t)$, the switching function for the first channel becomes a reference function, $F_2(t)$ is the same except for a time delay of T/N, $F_3(t)$ is delayed by $2T/N$, etc. Substitution of (15) in (3) gives the relations:

$$\left.\begin{aligned} A_{mj} &= A_{m1} \\ \theta_{mj} &= \theta_{m1} + (j - 1)2m\pi/N \end{aligned}\right\} \quad (16)$$

If we further suppose that the commutator makes contact between the typical channel and the line throughout a fraction x of the time interval

T/N allotted to that channel and breaks contact throughout the remainder of the switching cycle, we may write the reference switching function as:

$$F_1(t) = \begin{pmatrix} A, & -xT/2n < t < xT/2N \\ 0, & xT/2N < t < (2N - x)T/2N \end{pmatrix} \tag{17}$$

Hence from (4)

$$\left.\begin{aligned} A_{01} &= Ax/N, \\ A_{m1} &= \frac{2A}{m\pi} \sin \frac{mx\pi}{N}, \quad m > 0 \\ \theta_{m1} &= 0 \end{aligned}\right\} \tag{18}$$

In the receiving device, the corresponding switching process should be delayed with respect to the transmitter by a time interval t_0 equal to the time of transmission of the line. Hence we write

$$\left.\begin{aligned} B_{01} &= Bx/N \\ B_{m1} &= \frac{2B}{m\pi} \sin \frac{mx\pi}{N}, \quad m > 0 \\ \Phi_{m1} &= mqt_0 \end{aligned}\right\} \tag{19}$$

B_{mj} and θ_{mj} are related to B_{m1} and Φ_{m1} in a manner analogous to (16).

The time of transmission of a distorting line is not precisely definable, but may be represented for our purpose by a linear phase component of $Z(i\omega)$. That is, we write

$$Z(i\omega) = Z_0(i\omega)e^{-it_0\omega}, \tag{20}$$

where t_0 is the slope of a straight line giving the best linear approximation to the phase vs. frequency curve, and $Z_0(i\omega)$ is the impedance function remaining after the subtraction of $t_0\omega$ from the actual phase shift ordinates. Substituting (15)–(19) in (12), we find

$$\begin{aligned} Y_{jk} = ABe^{-it_0\omega_j} \Bigg(& \frac{x^2}{N^2} Z_0(i\omega_j) \\ & + \sum_{m=1}^{\infty} \frac{\sin^2 mx\pi/N}{m^2\pi^2} Z_0[i(mq + \omega_j)]e^{-i(j-k)2m\pi/N} \\ & + \sum_{m=1}^{\infty} \frac{\sin^2 mx\pi/N}{m^2\pi^2} Z_0^*[i(mq - \omega_j)]e^{i(j-k)2m\pi/N} \Bigg) \end{aligned} \tag{21}$$

If the attenuation of the line is constant throughout the range ω_j to $Mq + \omega_j$ and all frequencies above the latter value are suppressed, (21) becomes

$$Y_{jk} = \frac{ABx^2 Z_0 e^{-it_0\omega_j}}{N^2} \left[1 + 2 \sum_{m=1}^{M} \left(\frac{\sin mx\pi/N}{mx\pi/N} \right)^2 \cos (j - k) 2m\pi/N \right] \quad (22)$$

The crosstalk ratio or ratio of amplitude of signal received in the k^{th} channel to that received in the j^{th} channel when signal is transmitted in the j^{th} channel is, therefore,

$$\frac{Y_{jk}}{Y_{jj}} = \frac{1 + 2 \sum_{m=1}^{M} \left(\frac{\sin m\pi\, x/N}{m\pi\, x/N} \right)^2 \cos 2m\pi(k - j)/N}{1 + 2 \sum_{m=1}^{M} \left(\frac{\sin m\pi\, x/N}{m\pi\, x/N} \right)^2} \quad (23)$$

Results of calculations made for a 10-channel system from (23) for $x = 1$ and $x = .5$, corresponding to no lost time and half lost time respectively in switching are shown in Fig. 2. It may be noted that adjacent channel crosstalk with half lost time is equivalent to alternate channel crosstalk with no lost time. Examination of the curves reveals a number of significant facts, among which are:

1. Crosstalk is quite imperfectly suppressed when the band width of the line is smaller than the theoretical minimum—the width of one sideband multiplied by the number of channels.

2. As the band width of the line is increased above the theoretical minimum, improvement in crosstalk suppression increases slowly, so that in general the use of frequency range on the line is uneconomical compared with other systems. For example, with no lost time in switching, the band width of the line must be increased tenfold to suppress adjacent channel crosstalk by 40 *db*. This conclusion is, however, to be qualified as follows:

3. When the duration of contact is decreased (less of the available channel time used) definite optimum transmission band widths appear which give better crosstalk suppression than bands somewhat wider or narrower. This suggests the possibility of critical phase relations existing between the contributions from the various sidebands such that if the right number having proper amplitudes and phases can be combined, complete suppression of crosstalk may occur even when the transmitted band width is finite.

When x, the fraction of contact time used, is made to approach zero, the limit of the amplitude factor (18) for the typical harmonic of the switching function is $A_{m1} = 2Ax/N$, which is independent of m. This is consistent with the known fact that a wave consisting of periodically repeated sharp pulses is composed of a large number of harmonics of nearly equal

amplitude. If we use very short contact durations in time division, we should accordingly expect a large number of sidebands of nearly equal amplitude. The combination of proper numbers and phases of these sidebands offers a key to the realization of a time division multiplex system giving good crosstalk suppression with economy of frequency band.

Suppose that the duration of contact time is made sufficiently small to realize approximately the limiting values $A_{m1} = 2Ax/N$, $B_{m1} = 2Bx/N$ in transmitting and receiving respectively for tbe first $2M + 1$ of the sidebands and that by means of a low-pass filter with linear phase shift and

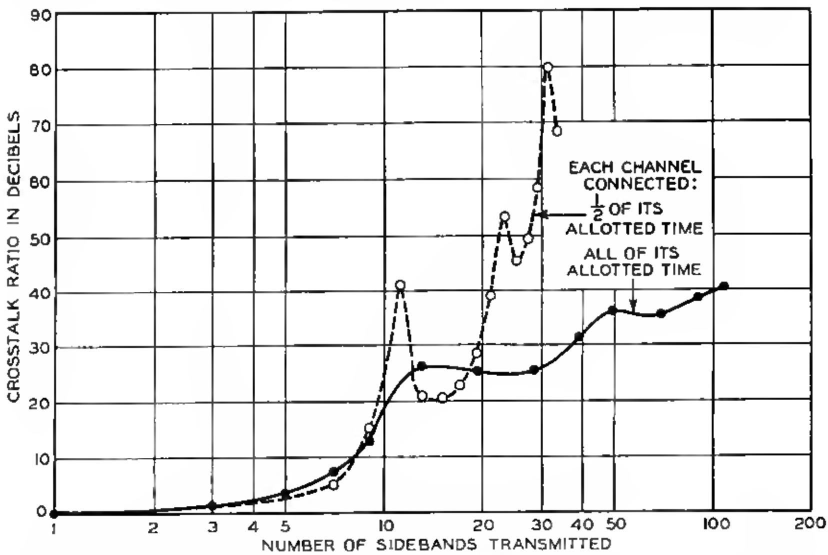

Fig. 2—Crosstalk between adjacent channels of ten-channel time division multiplex system with on-and-off switching. No attenuation or phase distortion in pass band of line

uniform attenuation in its pass-band all other sidebands are removed from the line. The expression (23) then becomes:

$$\frac{Y_{jk}}{Y_{jj}} = \frac{1 + 2 \sum_{m=1}^{M} \cos 2m\pi(k - j)/N}{1 + 2M} = \begin{cases} 1, & k = j \\ \dfrac{\sin (2M + 1)\pi(k - j)/N}{(2M + 1) \sin \pi(k - j)/N}, & k \neq j \end{cases} \tag{24}$$

In particular, if

it follows that

$$\left.\begin{aligned} M &= (N - 1)/2, \\ Y_{jk}/Y_{jj} &= 0, \; k \neq j \end{aligned}\right\} \tag{25}$$

Thus there exists in theory a system employing sidebands on zero frequency and the first $(N - 1)/2$ harmonics of the switching frequency, in which multichannel transmission is possible without interchannel interference. Since the required condition (25) may also be written $N = 2M + 1$, an odd number of channels is obtained. Since N sidebands are transmitted, the band width used is the same as the minimum required for N single sideband amplitude modulation channels on a frequency discrimination basis. Sidebands produced on higher harmonics in the time division process must be removed by filtering.

It is to be noted that since it is equality of the N sideband contributions which is important and the amount of each contribution is determined by the transmission characteristic of the line as well as the transmitting and receiving switching processes, it would be theoretically possible to make up for sideband irregularities by equalizing the line. However, the equalization required in the line would be of "stairstep" type rather than smoothly varying with frequency since an error in the value of one harmonic of the switching function produces the same error throughout the entire range occupied by the pair of sidebands associated with that harmonic.

General Switching Functions with Crosstalk Suppression and Minimum Band Width

The above discussion based on the properties of a commutator has led us to an ideal switching function which is, except for an unimportant proportionality factor,

$$F_j(t) = 1 + 2 \sum_{m=1}^{(N-1)/2} \cos m[qt - (j - 1)2\pi/N], \; N \text{ odd} \tag{26}$$

This type of switching is approximately realizable with synchronized commutators having contact widths very narrow in comparison with the spacing between contacts. For a 3000-cycle speech band, the minimum switching rate would be 6000 cycles per second. Such a speed would be difficult to obtain with ordinary mechanical means but would be feasible with rotating electron beams.

The concept of combining detected contributions from a number of sidebands in proper phase to give in-phase addition of desired components and cancellation of unwanted ones leads to a generalization of the switching processes over those possible with synchronized commutators. We note that the switching functions $F_j(t)$ of (2) and $G_k(t)$ of (9) are analogous to

carrier waves applied to a product modulator, and an electrical analogue of time division may be realized therefore by applying signal and a suitable carrier to a product modulator. Phase shifts in the carrier supply circuit may be made to serve the same purpose as the angular displacements between commutator segments. It is thus of interest to examine various other possible forms of the function $F_j(t)$ which are suitable for multiplex transmission and investigate methods by which they can be realized.

We note that (26) is suitable for an odd number of channels because it makes use of the direct signal component (or sideband on zero frequency) in addition to the paired sidebands on harmonics of the switching frequency. It seems reasonable to expect that systems for even numbers of channels can be devised using only upper and lower sidebands on harmonics and omitting the signal itself. Complete information for the separation of N channels should be contained in any set of N sidebands; hence we should not be forced to start with the sidebands of lowest frequency, but be able to use other sets with a more suitable place in the spectrum or with better equalization of amplitudes.

We shall derive an expression for a quite general switching function meeting the desired conditions of freedom from crosstalk and economy of band width for an even number of channels by assuming the following forms for A_{mj} and θ_{mj} in (3),

$$A_{mj} = \begin{pmatrix} A,\ n \leqq m \leqq n + N/2 - 1 \\ 0,\ m < n, \text{ or } m > n + N/2 - 1 \end{pmatrix} \tag{27}$$

$$\theta_{mj} = (j - 1)\,(m + h)\,\psi + \alpha \tag{28}$$

The switching function assumed contains $N/2$ harmonics and hence will produce N sidebands. The values of n, h, α and ψ are first assumed to be arbitrary. At the receiving end, a switching function similar except for a time displacement t_0 will be assumed. That is, in (9), we take

$$B_{mk} = \begin{pmatrix} B,\ n \leqq m \leqq n + N/2 - 1 \\ 0,\ m < n \text{ or } m > n + N/2 - 1 \end{pmatrix} \tag{29}$$

$$\Phi_{mk} = (k - 1)\,(m + h)\psi + (mq + \nu)\tau + \alpha \tag{30}$$

Transmission over the line is assumed to be of the distortionless form obtained by setting $Z_0(i\omega) = Z_0$, a constant, in (20). Substituting (27)–(30) in (14), we then calculate

$$Y_{jk} = \frac{NABZ_0e^{-it_0\omega_j}}{4}\begin{cases} 1, & j = k \\ \dfrac{2 \sin N(j-k)\psi/4 \cos (4n+4h+N-2)(j-k)\psi/4}{N \sin (j-k)\psi/2}, & j \neq k \end{cases} \tag{31}$$

Figure 3 shows the curve of admittance vs. frequency required for on-and-off and plus-and-minus switching. Referred to the mid-band admittance as unity, the admittance is reduced to one-half (six *db* loss) at the frequencies $rNq/2$ and $(r + 2)Nq/2$ which are the nominal upper and lower cutoffs. N is an even integer and r is zero or any positive integer. The admittance curve has odd-symmetry about the cutoff frequencies—that is, if at a frequency x cycles below a cutoff frequency, the admittance has the value a, it must be $1 - a$ at a frequency x cycles above the cutoff. The nominal

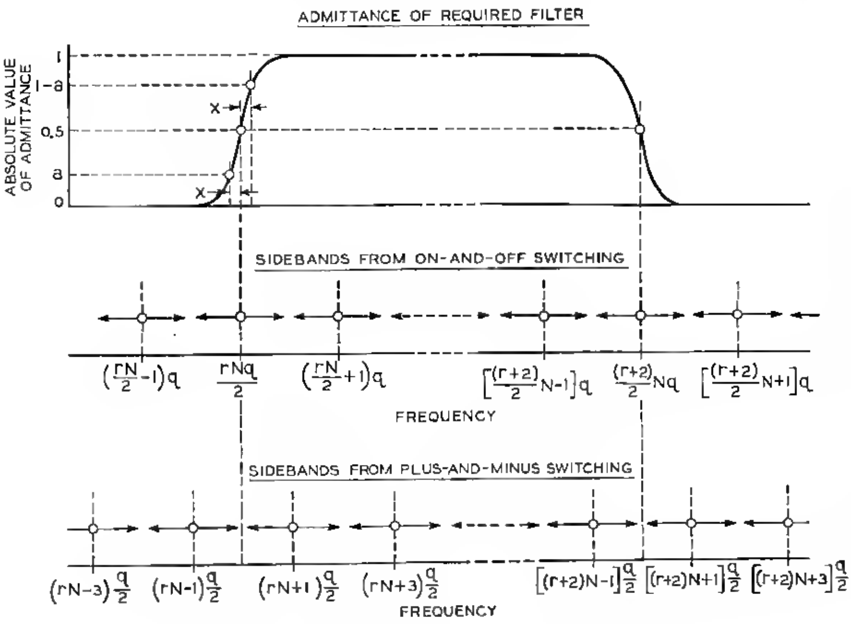

Fig. 3—Vestigial sideband transmission in time division multiplex systems

band transmitted in the case of on-and-off switching consists of the upper sideband on the harmonic $rNq/2$, the lower sideband on the harmonic $(r + 2)Nq/2$, and all intervening sidebands. In the case of plus-and-minus switching the upper and lower sidebands on frequencies $(rN + 1)q/2$ to $[(r + 2)N - 1]q/2$ inclusive are transmitted. Impairment of the nominal band by the filter is made up by transmitting the appropriate parts of sidebands outside the nominal range. It is easily verified that either of the systems depicted in Fig. 3 satisfies the required conditions for multiplex transmission without interchannel interference when the sidebands produced by a given signal have equal amplitudes over the range utilized.

The vestigial method is required only when strong sidebands very near the desired ones must be removed. Modifications of the time division process exist in which vestigial filters are unnecessary because very little energy appears at unwanted side frequencies. It is clear that if we regard the problem as one of producing certain sidebands on carriers of definite phases, we are not restricted to commutating devices only but may make use of general modulator technique. Further details concerning specific circuit arrangements are described in *U. S. Patent* 2,213,938, W. R. Bennett; and *U. S. Patent* 2,213,941, E. Peterson. As a general guide the following table of carrier phases for an N-channel system (N even) is furnished:

Table of Phase Shifts for N-Channel System

($N/2$ Carrier Frequencies Required)

Carrier Frequency		ν	$\nu + q$	$\nu + 2q$	$\nu + 3q$	$\cdots$
Phase Shift In Carrier	Channel 1	0	0	0	0	$\cdots$
	2	π/N	$3\pi/N$	$5\pi/N$	$7\pi/N$	$\cdots$
	3	$2\pi/N$	$6\pi/N$	$10\pi/N$	$14\pi/N$	$\cdots$
	4	$3\pi/N$	$9\pi/N$	$15\pi/N$	$21\pi/N$	$\cdots$

Transmission Requirements

Practical success of a time division multiplex system requires the maintenance of a satisfactory ratio of wanted signal to crosstalk. In order to accomplish this, the transmission link must preserve the amplitude and phase relations of a group of sidebands. A physical picture of the relations involved may be obtained from Fig. 4, which is drawn for the particular case of a 5-channel system of the on-and-off switching type. For this example the theory previously developed shows that five sidebands of equal amplitude are sufficient, namely—the signal itself (which may be regarded as a sideband on a carrier of zero frequency), the upper and lower sidebands on the switching frequency and on the second harmonic of the switching frequency. If we take the phases of the switching fundamental and its second harmonic as applied to the first channel as a reference, the proper phases of fundamental and second harmonic respectively for the other four channels are given by the following table:

Channel Number	Fundamental Phase	Second Harmonic Phase
2	72°	144°
3	144°	288°
4	216°	432°
5	288°	576°

In Fig. 4, we have assumed a single-frequency signal component as input to the first channel. If the line has distortionless attenuation and phase characteristics, the five resulting side frequencies are received in the first channel as the five in-phase vectors of equal amplitude shown in (a). Re-

ception in the second channel is shown by (*b*) in which the vector 1 represents the directly transmitted signal component (or sideband on *d-c*), 2 and 3

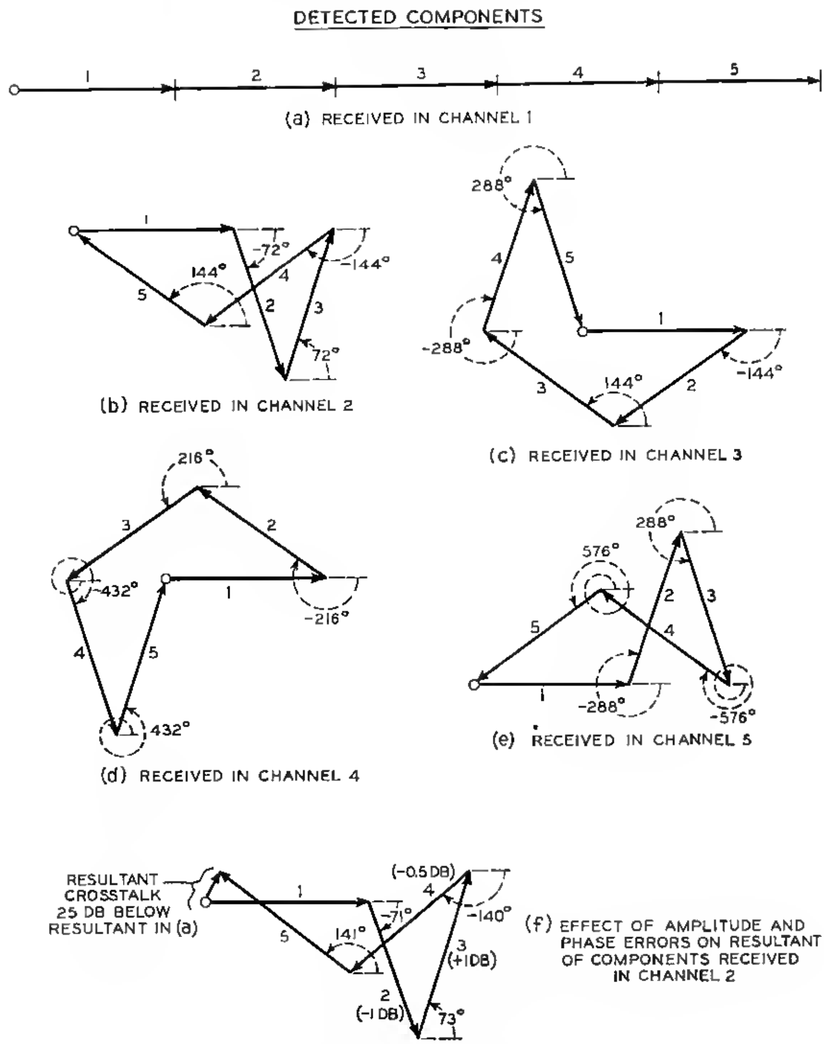

Fig. 4—Graphical representation of operation of time division multiplex system. Signal transmitted in channel 1 of 5-channel 5-sideband system

represent the detected components from the upper and lower side frequencies associated with the fundamental switching frequency, and 4 and 5 the components resulting from the upper and lower side frequencies of the

second harmonic. Components 2 and 3 are shifted $-72°$ and $+72°$ respectively and 4 and 5 are shifted $-144°$ and $+144°$ in relation to the phase of component 1. As shown in (*b*), the five vectors combine in the form of a closed polygon giving a resultant of zero amplitude. Similar vector diagrams for reception in the third, fourth, and fifth channels are shown in (*c*), (*d*), and (*e*). The appropriate diagrams for transmission in channels 2, 3, 4, and 5 and receiving in any channel can be obtained from (*a*) — (*e*) by cyclic permutation of the channel numbers, i.e., transmission in 1 and reception in 2 corresponds to transmission in 2 and reception in 3, etc.

Production of crosstalk by phase and amplitude distortion in the transmission medium is illustrated by (*f*), Fig. (4), which shows the resultant component received in channel 2 when signal is transmitted in channel 1 and an imperfect line is used to connect the transmitting and receiving terminals. The vector 1 is taken as the reference amplitude and phase. The gain characteristic of the line is assumed to be one *db* low at the side frequency producing vector 2, one *db* too high for vector 3, 0.5 *db* low for vector 4, and with no error for vector 5. The phase curve is assumed to depart from a straight line by $-1°$, $-1°$, $-4°$, $+3°$ at the side frequencies from which components 2, 3, 4, 5 respectively are derived. The vector polygon fails to close and the resultant represents an unwanted signal received in channel 2 at a level 25 *db* below the wanted signal received in channel 1.

We may make an estimate of the accuracy of the equalization required in the general case by writing the transfer impedance $Z(i\omega)$ in the form:

$$Z(i\omega) = \rho(\omega) Z_0 e^{-it_0\omega - i\beta(\omega)} \tag{39}$$

where $\beta(\omega)$ represents the departure of the phase shift from a straight line and the variation from flat gain is given by

$$g(\omega) = 20 \log_{10} \rho(\omega) \tag{40}$$

The expression (39) may be rewritten as:

$$Z(i\omega) = [1 + z(i\omega)] Z_0 e^{-it_0\omega}, \tag{41}$$

where

$$z(i\omega) = \rho(\omega) e^{-i\beta(\omega)} - 1 \tag{42}$$

If we assume that the switching function is of the general form (34), we calculate from (14) the general relation:

$$Y_{jk} = K \begin{cases} N + \sum_{m=n}^{n+\frac{N}{2}-1} (z[i(\nu + mq + \omega_j)] + z^*[i(\nu + mq - \omega_j)]), \; j = k \\ \sum_{m=n}^{n+\frac{N}{2}-1} (z[i(\nu + mq + \omega_j)]e^{-i(j-k)(2m-2n+1)\pi/N} \\ \qquad + z^*[i(\nu + mq - \omega_j)]e^{i(j-k)(2m-2n+1)\pi/N}), \; j \neq k \end{cases} \tag{43}$$

where K is a complex constant of proportionality. The case of $j = k$ which gives transmission within the channel contains a variation with signal frequency caused by the summation of the departures from ideal transmission at the N sideband frequencies. This term presumably will be unimportant if the transmission characteristic is sufficiently good to meet crosstalk requirements; hence we may neglect the z and z^* terms in the case of $j = k$ and write the ratio of interference to desired signal as:

$$C_{jk} = \frac{Y_{jk}}{Y_{jj}} = \frac{1}{N} \sum_{m=n}^{n+\frac{N}{2}-1} (z[i(\nu + mq + \omega_j)]e^{-i(j-k)(2m-2n+1)\pi/N} + z^*[i(\nu + mq - \omega_j)]e^{i(j-k)(2m-2n+1)\pi/N}) \tag{44}$$

The crosstalk ratio will in general vary with the signal frequency. The requirement would logically be based on the total interference power weighted in accordance with the interfering effect at individual frequencies. Thus we might set

$$X_{jk} = S_j \int_{\omega_a}^{\omega_b} W_{jk}(\omega_j) \,|\, C_{jk} \,|^2 \, d\omega_j, \tag{45}$$

where X_{jk} represents the weighted interference power received in the k^{th} channel when a reference signal wave of mean power S_j is applied to the j^{th} channel. The limits of integration ω_a and ω_b are the lowest and highest signal frequencies used. The function $W_{jk}(\omega_j)$ represents the proper weighting with frequency of the interference and takes into account the distribution of the interfering signal and the relative importance of the different interfering frequencies.

Equation (45) is sufficient for computation of interchannel interference introduced by the line when the transmission characteristics of the line are known. A more valuable result, however, would be the expression of the required line characteristics in terms of the allowable interference. In general this would require some specification of the nature of the departures from the ideal characteristic. Except perhaps for systems with very few channels, it seems reasonable to assume that the departures are distributed fairly uniformly throughout the frequency range transmitted by the line,

and hence that for purposes of estimating requirements we may replace $|C_{jk}|^2$ in (45) by its average value over the band. We may then write (45) in the form:

$$\overline{|C_{jk}|^2} = \frac{X_{jk}}{S_j \int_{\omega_a}^{\omega_b} W_{jk}(\omega_j)\, d\omega_j} \equiv U_{jk} \tag{46}$$

The value of the right-hand member, which we have designated by the symbol U_{jk}, is either known or can be determined for the particular type of signal. Hence our problem is reduced to finding the allowable departures in transmission which keep the mean square absolute value of C_{jk} from exceeding a prescribed maximum value.

We note that C_{jk} is the sum of N complex quantities, each of which is restricted to a range of values determined by the transfer impedance of the line in an individual band of frequencies. A convenient simplification may be made by regarding the N complex quantities as N independent chance variables. This is tantamount to assuming that departures in one band do not affect departures in any other band; the assumption is not strictly true, but should lead to no important error. We may then make use of the following theorem[8]: If

$$\zeta = b_1 z_1 + b_2 z_2 + \cdots + b_n z_n, \tag{47}$$

where $z_1, z_2, \cdots z_n$ are n independent complex chance variables and $b_1, b_2, \cdots b_n$ are complex constants,

$$\overline{|\zeta|^2} = \overline{|b_1|^2}\,\overline{|z_1|^2} + \cdots + \overline{|b_n|^2}\,\overline{|z_n|^2} \tag{48}$$

Application of this theorem to (44) gives

$$\begin{aligned}\overline{|C_{jk}|^2} &= \frac{1}{N^2} \sum_{m=n}^{n+\frac{N}{2}-1} \left(\overline{|z[i(\nu + mq + \omega_j)]|^2} + \overline{|z^*[i(\nu + mq - \omega_j)]|^2}\right) \\ &= \overline{|z|^2}/N, \end{aligned} \tag{49}$$

if the average square of the absolute value of the departure is the same in all bands and is equal to $\overline{|z|^2}$, which we shall define as the average squared absolute value of the departure for the entire line band used.

From (42) and (40),

$$\begin{aligned}|z(i\omega)|^2 &= 1 - 2\rho(\omega)\cos\theta(\omega) + \rho^2(\omega) \\ &= 1 - 2\cdot 10^{g(\omega)/20}\cos\theta(\omega) + 10^{g(\omega)/10}\end{aligned} \tag{50}$$

[8] R. S. Hoyt, *B. S. T. J.*, Vol. XII, No. 1, Jan. 1933, p. 64.

Since it seems certain that g and θ must remain small to make the system operative, we investigate the nature of (50) when expanded in powers of g and θ. The leading terms are:

$$|z(i\omega)|^2 = \frac{(\log_e 10)^2}{400} g^2(\omega) + \theta^2(\omega) + \cdots \tag{51}$$

Hence for g and θ small, we have independent of any correlation which may exist between g and θ,[9]

$$\overline{|z(i\omega)|^2} = \left(\frac{\log_e 10}{20}\right)^2 \overline{g^2(\omega)} + \overline{\theta^2(\omega)} \tag{52}$$

Let

$$\overline{\sigma_1^2} = \overline{g^2(\omega)}, \qquad \overline{\sigma_2^2} = \overline{\theta^2(\omega)} \tag{53}$$

Then from (46), (49), (52),

$$U_{jk} = \left[\left(\frac{\log_e 10}{20}\right)^2 \sigma_1^2 + \sigma_2^2\right] \Big/ N \tag{54}$$

In (54) σ_1 is the r.m.s. departure of the gain in *db* from a constant and σ_2 is the r.m.s. departure of the phase shift in radians from a straight line. If σ_2 is expressed in degress instead of radians, (54) becomes

$$U_{jk} = 10^{-3}\,(13.25\,\sigma_1^2 + .3046\,\sigma_2^2)/N \tag{55}$$

The total interference received in any one channel is the sum of the individual contributions from the other $N - 1$ channels. The addition factor required to express the total in terms of the interference from one channel depends on the nature of the individual loads. Thus if the probability that any one channel is transmitting a signal wave is τ, the average total interference power received in one channel is

$$X = \tau(N - 1)X_{jk} = US_j \int_{\omega_a}^{\omega_b} W_{jk}(\omega_j)\, d\omega_j, \tag{56}$$

where

$$U = (N - 1)\tau U_{jk} = \frac{(N - 1)\tau}{N} 10^{-3}(13.25\sigma_1^2 + .3046\sigma_2^2) \tag{57}$$

For large values of N, the ratio $(N - 1)/N$ approaches unity, and the average interference becomes independent of the number of channels. The average interference may not be the most significant quantity, however. For example, if there is a considerable probability that all channels are

[9] This method of avoiding any assumption concerning correlation of attenuation and phase was suggested by Dr. T. C. Fry.

carrying energy simultaneously, as would be the case if the channels were subdivisions of a common signal band, the peak value of interference would probably be of more significance than the average value.

It is convenient to let

$$H = 10 \log_{10} \frac{S_j}{X} \tag{58}$$

$$F = -10 \log_{10} \int_{\omega_a}^{\omega_b} W_{jk}(\omega_j)\, d\omega_j \tag{59}$$

H is the ratio expressed in db of mean signal power in one channel to the total interference power received in one channel, and F is the weighting factor expressed in db. From (56),

$$U = 10^{-(H-F)/10} \tag{60}$$

Equation (57) may be written in the form,

$$\frac{\sigma_1^2}{a^2} + \frac{\sigma_2^2}{b^2} = 1, \tag{61}$$

where

$$\left.\begin{aligned} a &= 8.69 \sqrt{\frac{NU}{(N-1)\tau}}\ db \\ b &= 57.3 \sqrt{\frac{NU}{(N-1)\tau}}\ \text{degrees} \end{aligned}\right\} \tag{62}$$

Without the numerical factors, a and b are expressed in nepers and radians respectively.

If we regard σ_1 and σ_2 as variables, (61) determines a family of ellipses in which a and b are the semi-axes. By assigning values to N, τ, and $H - F$ we may thus represent the requirements on gain and phase variation by elliptical boundaries in the $\sigma_1\sigma_2$-plane. Figure 5 shows such a diagram constructed for a large number of channels each active one-fourth of the time and with flat weighting. In terms of the symbols above, we have set $N/(N-1)$ equal to unity, $\tau = 1/4$, and $F = 0$. Gain and phase variations included within a particular ellipse produce average interference power less than the amount designated on the boundary in terms of db down on mean power in one channel. The requirements imposed on both gain and phase variation are considerably more stringent than the corresponding requirements for carrier systems using frequency discrimination and employing comparable band widths.

Requirements on linear transmission of the line are, of course, not the only considerations involved in a comparison of time division multiplex

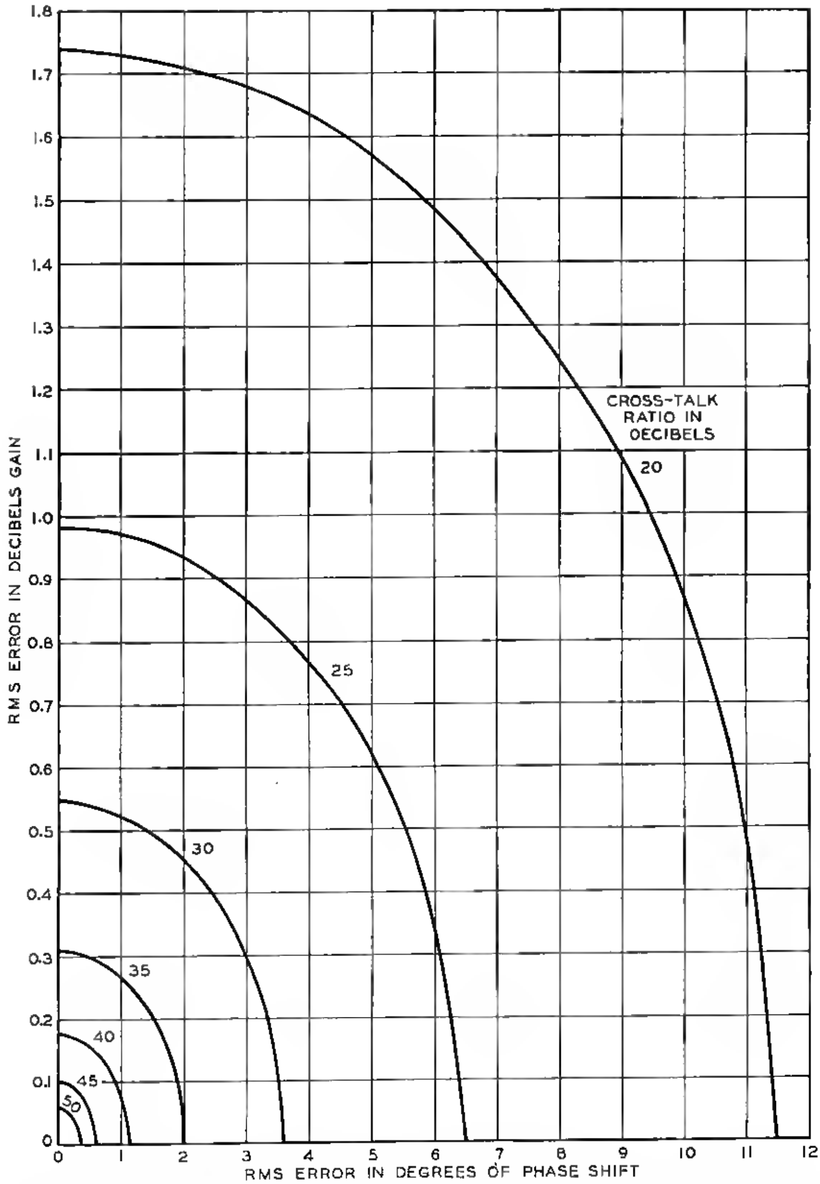

Fig. 5—Gain and phase requirements for transmission of time division multiplex signals. Each channel active 25% of time

methods with competing methods of superimposing channels. Other aspects to be considered are the synchronization of transmitting and receiving switching processes, the effects of non-linearities in the line, and the sensitivity of the system to external interference. It is thought, however, that the severe restrictions imposed on phase and attenuation characteristics when economy of band width is required form the weakest feature of the method and will in many cases provide the primary criterion for judging its availability in the solution of particular problems. Conversely, if the crosstalk requirements of the system are sufficiently mild to enable the transmission problem to be solved, the other problems also become relatively simplified.

Further References

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